

Imitation Learning with Stability and Safety Guarantees

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Abstract—A method is presented to synthesize neural network (NN) controllers with stability and safety guarantees through imitation learning. Convex stability and safety conditions are derived for linear time-invariant plant dynamics with NN controllers. The proposed approach merges Lyapunov theory with local quadratic constraints to bound the nonlinear activation functions in the NN. The safe imitation learning problem is formulated as an optimization problem with the goal of minimizing the imitation learning loss, and maximizing volume of the region of attraction associated with the NN controller, while enforcing the stability and safety conditions. An alternating direction method of multipliers based algorithm is proposed to solve the optimization. The method is illustrated on an inverted pendulum system and aircraft longitudinal dynamics.

I. INTRODUCTION

Imitation learning (IL) is a class of methods that learns a policy to attain a control goal consistent with expert demonstrations [1], [2]. Used in tandem with deep neural networks (NN), IL presents unique advantages, including a substantial increase in sample efficiency compared to reinforcement learning (RL) [3], and wide applicability to domains where the reward model is not accessible or on-policy data is difficult/unsafe to acquire [1]. While IL is closely related to supervised learning as it trains a mapping from observations to actions [4], a key difference is the ensuing deployment of the learned policy under dynamics, which consequently raises the issue of closed-loop stability. This problem naturally falls within the realm of robust control, which provides rigorous analysis of stability for linear or nonlinear systems [5]; however, a major technical challenge is to derive nonconservative guarantees for highly nonlinear policies such as neural networks that can be also tractably incorporated into the learning process.

The paper tackles this issue and presents a method to synthesize NN controllers with stability and safety guarantees through IL. We first derive convex stability and safety conditions by merging Lyapunov theory with local (sector) quadratic constraints to describe the nonlinear activation functions in the NN. Then we formulate an optimization problem that minimizes the IL loss, and maximizes the region of attraction inner-estimates certified by the learned NN controller, while enforcing the stability and safety constraints. Finally, we propose an alternating direction method of multipliers (ADMM) based method to solve the optimization problem.

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The method of using quadratic constraints to characterize the nonlinear activation functions has been proposed in [6], and applied to Lipschitz constant estimation for NNs [7], forward reachability analysis of NN controlled systems [8], and local stability analysis of NN controlled uncertain systems [9], [10]. Reference [11] proposes an ADMM based robust NN training method that minimizes the training loss and the Lipschitz constant simultaneously. Reference [12] formulates convex sets of recurrent NN with bounded incremental ℓ_2 gain using incremental quadratic constraints. The work [13] performs global stability analysis of RL-controlled nonlinear systems based on integral quadratic constraint theory.

Compared to existing works, our approach makes the following contributions. First, to the best of our knowledge, this is the first attempt to ensure stability of IL based on deep neural networks. The stability condition from [9] is nonconvex and thus computationally intractable for NN control synthesis; here we convexify this constraint (using loop transformation) for its efficient enforcement in the learning process. Notably, a well-known challenge in IL is the existence of suboptimal demonstrations, which can even come from unstable controllers. As demonstrated in the case studies, while the proposed approach can train a policy that imitates the expert demonstrations, it can potentially improve local stability over suboptimal expert policies, thus enhance the robustness of IL.

The stability and safety issues for learning (especially RL) based control has also been addressed by a Hamilton-Jacobi reachability-based framework in [14], a control barrier function based method in [15], [16], and a convex projection operator based method in [17].

The paper is organized as follows. Section II presents the problem formulation. Section III reviews the stability condition proposed in [9], and Section IV presents a loop transformation based method to convexify it. Section V formulates the safe IL problem as an optimization problem, and proposes an ADMM based method to solve it. Section VI provides numerical examples, including an inverted pendulum example, and an aircraft example. Section VII summarizes the results.

Notation: \mathbb{S}^n denotes the set of n -by- n symmetric matrices. \mathbb{S}_+^n and \mathbb{S}_{++}^n denote the sets of n -by- n symmetric, positive semidefinite and positive definite matrices, respectively. For $v, w \in \mathbb{R}^n$, the inequality $v \leq w$ is interpreted elementwise, i.e. $v_i \leq w_i$ for $i = 1, \dots, n$. Similarly, $v \geq w$ is an elementwise inequality. For $P \in \mathbb{S}_{++}^n$, define the ellipsoid

$$\mathcal{E}(P) := \{x \in \mathbb{R}^n : x^\top P x \leq 1\}. \quad (1)$$

II. PROBLEM FORMULATION

Consider the feedback system consisting of a plant G and state-feedback controller π as shown in Figure 1. We assume

the plant G is a linear, time-invariant (LTI) system defined by the following discrete-time model:

$$x(k+1) = A_G x(k) + B_G u(k), \quad (2)$$

where $x(k) \in \mathbb{R}^{n_G}$ is the state and $u(k) \in \mathbb{R}^{n_u}$ is the control. The state matrices have compatible dimensions, i.e. $A_G \in \mathbb{R}^{n_G \times n_G}$ and $B_G \in \mathbb{R}^{n_G \times n_u}$. Finally, assume $x(k)$ is constrained to a set $X \subset \mathbb{R}^{n_G}$, which is referred to as the “safety condition”. This state constraint set is assumed to be a polytope symmetric around the origin:

$$X = \{x \in \mathbb{R}^{n_G} : -h \leq Hx \leq h, \ h \geq 0\}, \quad (3)$$

where $H \in \mathbb{R}^{n_x \times n_G}$, and $h \in \mathbb{R}^{n_x}$. The controller $\pi : \mathbb{R}^{n_G} \rightarrow \mathbb{R}^{n_u}$ is an ℓ -layer, feedforward neural network (NN) defined as:

$$w^0(k) = x(k), \quad (4a)$$

$$w^i(k) = \phi^i(W^i w^{i-1}(k) + b^i), \ i = 1, \dots, \ell, \quad (4b)$$

$$u(k) = W^{\ell+1} w^\ell(k) + b^{\ell+1}, \quad (4c)$$

where $w^i \in \mathbb{R}^{n_i}$ are the outputs (activations) from the i^{th} layer and $n_0 = n_G$. The operations for each layer are defined by a weight matrix $W^i \in \mathbb{R}^{n_i \times n_{i-1}}$, bias vector $b^i \in \mathbb{R}^{n_i}$, and activation function $\phi^i : \mathbb{R}^{n_i} \rightarrow \mathbb{R}^{n_i}$. The activation function ϕ^i is applied element-wise, i.e.

$$\phi^i(v) := [\varphi(v_1), \dots, \varphi(v_{n_i})]^\top, \quad (5)$$

where $\varphi : \mathbb{R} \rightarrow \mathbb{R}$ is the (scalar) activation function selected for the NN. Common choices for the scalar activation function include $\varphi(v) := \tanh(v)$, sigmoid $\varphi(v) := \frac{1}{1+e^{-v}}$, ReLU $\varphi(v) := \max(0, v)$, and leaky ReLU $\varphi(v) := \max(av, v)$ with $a \in (0, 1)$. We assume the activation φ is identical in all layers; this can be relaxed with minor changes to the notation.

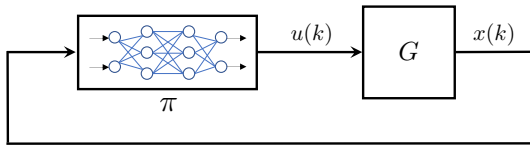


Fig. 1: Feedback system with plant G and NN π

Remark 1. Note that the control constraint is not considered in this paper. However, if the control constraint set is a hypercube, it can be addressed by scaling the dynamics (2) so that the control constraint set becomes $[-1_{n_u \times 1}, 1_{n_u \times 1}]$, and applying \tanh elementwise to the output layer (4c).

Assumption 1. We assume the equilibrium point of the closed-loop system is at the origin $x_* = 0_{n_G \times 1}$ with input $u_* = 0_{n_u \times 1}$. To ensure this assumption holds, we restrict ourselves to zero bias terms: $b^i = 0_{n_i \times 1}$, for $i = 1, \dots, \ell + 1$, and we also assume that the activation φ satisfies $\varphi(0) = 0$.

Let $\chi(k; x_0)$ denote the solution to the feedback system at time k from initial condition $x(0) = x_0$. The region of attraction (ROA) is defined below.

Definition 1. The region of attraction (ROA) of the feedback system with plant G and NN π is defined as

$$\mathcal{R} := \{x_0 \in X : \lim_{k \rightarrow \infty} \chi(k; x_0) = 0_{n_G \times 1}\}. \quad (6)$$

Given state and control data pairs from the expert demonstrations, our goal is to learn a NN controller from the data to reproduce the demonstrated behavior, while guaranteeing the system trajectories under the control of the learned NN controller satisfy the safety condition ($x(k) \in X \ \forall k \geq 0$), and are able to converge to the equilibrium point if they start from the ROA associated with the learned NN controller.

III. STABILITY CONDITION FOR NN CONTROLLED SYSTEMS

In this section, we treat the parameters of the NN controller as fixed and analyze the safety and local stability conditions of the closed loop systems.

A. NN Representation: Isolation of Nonlinearities

It is useful to isolate the nonlinear activation functions from the linear operations of the NN as done in [6], [18]. Define v^i as the input to the activation function ϕ^i (recalling that $b^i = 0$ by Assumption 1):

$$v^i(k) := W^i w^{i-1}(k), \ i = 1, \dots, \ell. \quad (7)$$

The nonlinear operation of the i^{th} layer (Equation 4b) is thus expressed as $w^i(k) = \phi^i(v^i(k))$. Gather the inputs and outputs of all activation functions:

$$v_\phi := \begin{bmatrix} v^1 \\ \vdots \\ v^\ell \end{bmatrix} \in \mathbb{R}^{n_\phi} \text{ and } w_\phi := \begin{bmatrix} w^1 \\ \vdots \\ w^\ell \end{bmatrix} \in \mathbb{R}^{n_\phi}, \quad (8)$$

where $n_\phi := n_1 + \dots + n_\ell$, and define the combined nonlinearity $\phi : \mathbb{R}^{n_\phi} \rightarrow \mathbb{R}^{n_\phi}$ by stacking the activation functions:

$$\phi(v_\phi) := \begin{bmatrix} \phi^1(v^1) \\ \vdots \\ \phi^\ell(v^\ell) \end{bmatrix}. \quad (9)$$

Thus $w_\phi(k) = \phi(v_\phi(k))$, where the scalar activation function φ is applied element-wise to each entry of v_ϕ . Finally, using $b^i = 0_{n_i \times 1}$ for $i = 1, \dots, \ell + 1$, the NN control policy π defined in Equation 4 can be rewritten as:

$$\begin{bmatrix} u(k) \\ v_\phi(k) \end{bmatrix} = N \begin{bmatrix} x(k) \\ w_\phi(k) \end{bmatrix} \quad (10a)$$

$$w_\phi(k) = \phi(v_\phi(k)). \quad (10b)$$

The matrix N depends on the weights as follows, where the vertical and horizontal bars partition N compatibly with the inputs (x, w_ϕ) and outputs (u, v_ϕ) :

$$N := \left[\begin{array}{c|ccc|c} 0 & 0 & 0 & \dots & W^{\ell+1} \\ \hline W^1 & 0 & \dots & 0 & 0 \\ 0 & W^2 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & W^\ell & 0 \end{array} \right] \quad (11a)$$

$$:= \begin{bmatrix} N_{ux} & N_{uw} \\ N_{vx} & N_{vw} \end{bmatrix}. \quad (11b)$$

This decomposition of the NN, depicted in Figure 2, isolates the activation functions in preparation for the stability analysis.

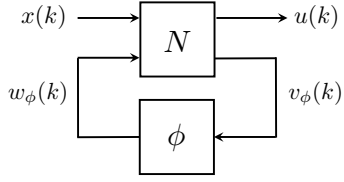


Fig. 2: NN representation to isolate the nonlinearities ϕ .

The equilibrium state $x_* = 0_{n_G \times 1}$ can be propagated through the NN. This yields the equilibrium values $v_* = w_* = 0_{n_\phi \times 1}$ as well as the equilibrium control command $u_* = 0_{n_u \times 1}$.

B. Quadratic Constraints: Scalar Activation Functions

The stability analysis relies on quadratic constraints to bound the activation function. A typical constraint is the sector bound as defined next.

Definition 2. Let $\alpha \leq \beta$ be given. The function $\varphi : \mathbb{R} \rightarrow \mathbb{R}$ lies in the (global) sector $[\alpha, \beta]$ if:

$$(\varphi(\nu) - \alpha\nu) \cdot (\beta\nu - \varphi(\nu)) \geq 0 \quad \forall \nu \in \mathbb{R}. \quad (12)$$

The interpretation of the sector $[\alpha, \beta]$ is that φ lies between lines passing through the origin with slope α and β . Many activation functions are bounded in the sector $[0, 1]$, e.g. \tanh and ReLU. Figure 3 illustrates $\varphi(\nu) = \tanh(\nu)$ (blue solid) and the global sector defined by $[0, 1]$ (red solid lines).

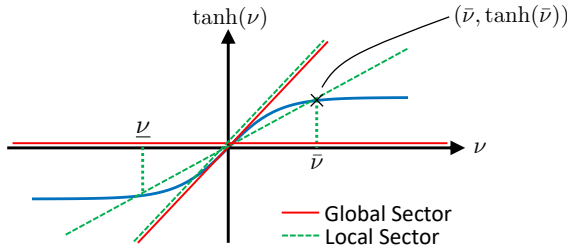


Fig. 3: Sector constraints on \tanh

The global sector constraint is often too coarse for analysis; thus, we consider a local sector constraint for tighter bounds.

Definition 3. Let $\alpha, \beta, \underline{\nu}, \bar{\nu} \in \mathbb{R}$ with $\alpha \leq \beta$ and $\underline{\nu} \leq 0 \leq \bar{\nu}$. The function $\varphi : \mathbb{R} \rightarrow \mathbb{R}$ satisfies the local sector $[\alpha, \beta]$ if

$$(\varphi(\nu) - \alpha\nu) \cdot (\beta\nu - \varphi(\nu)) \geq 0 \quad \forall \nu \in [\underline{\nu}, \bar{\nu}]. \quad (13)$$

As an example, $\varphi(\nu) := \tanh(\nu)$ restricted to the interval $[-\bar{\nu}, \bar{\nu}]$ satisfies the local sector bound $[\alpha, \beta]$ with $\alpha := \tanh(\bar{\nu})/\bar{\nu} > 0$ and $\beta := 1$. As shown in Figure 3 (green dashed lines), the local sector provides a tighter bound than the global sector. These bounds are valid for a symmetric interval around the origin with $\underline{\nu} = -\bar{\nu}$; non-symmetric intervals ($\underline{\nu} \neq -\bar{\nu}$) can be handled similarly.

C. Quadratic Constraints: Combined Activation Functions

Local sector constraints can also be defined for the combined nonlinearity ϕ , given by (9). Let $\underline{v}, \bar{v} \in \mathbb{R}^{n_\phi}$ be given with $\underline{v} \leq \bar{v}$. Suppose that the activation input $v_\phi \in \mathbb{R}^{n_\phi}$ lies element-wise in the interval $[\underline{v}, \bar{v}]$ and the i^{th} input/output pair is $w_i = \varphi(v_i)$, and the scalar activation function satisfies the local sector $[\alpha_i, \beta_i]$ with the input restricted to $v_i \in [\underline{v}_i, \bar{v}_i]$ for $i = 1, \dots, n_\phi$. The local sector bounds can be computed for φ on the given interval analytically (as above for \tanh). These local sectors can be stacked into vectors $\alpha_\phi, \beta_\phi \in \mathbb{R}^{n_\phi}$ that provide quadratic constraints satisfied by the combined nonlinearity ϕ .

Lemma 1. Let $\alpha_\phi, \beta_\phi, \underline{v}, \bar{v} \in \mathbb{R}^{n_\phi}$ be given with $\alpha_\phi \leq \beta_\phi$, and $\underline{v} \leq \bar{v}$. Suppose that ϕ satisfies the local sector $[\alpha_\phi, \beta_\phi]$ element-wise for all $v_\phi \in [\underline{v}, \bar{v}]$. For any $\lambda \in \mathbb{R}^{n_\phi}$ with $\lambda \geq 0$, it holds that

$$\begin{bmatrix} v_\phi \\ w_\phi \end{bmatrix}^\top \begin{bmatrix} -2A_\phi B_\phi \Lambda & (A_\phi + B_\phi) \Lambda \\ (A_\phi + B_\phi) \Lambda & -2\Lambda \end{bmatrix} \begin{bmatrix} v_\phi \\ w_\phi \end{bmatrix} \geq 0 \quad \forall v_\phi \in [\underline{v}, \bar{v}], w_\phi = \phi(v_\phi), \quad (14)$$

where $A_\phi = \text{diag}(\alpha_\phi)$, $B_\phi = \text{diag}(\beta_\phi)$, and $\Lambda = \text{diag}(\lambda)$.

Proof. For any $v_\phi \in \mathbb{R}^{n_\phi}$ and $w_\phi = \phi(v_\phi)$:

$$\begin{aligned} & \begin{bmatrix} v_\phi \\ w_\phi \end{bmatrix}^\top \begin{bmatrix} -2A_\phi B_\phi \Lambda & (A_\phi + B_\phi) \Lambda \\ (A_\phi + B_\phi) \Lambda & -2\Lambda \end{bmatrix} \begin{bmatrix} v_\phi \\ w_\phi \end{bmatrix} \\ &= \sum_{i=1}^{n_\phi} \lambda_i (w_i - \alpha_i v_i) \cdot (\beta_i v_i - w_i). \end{aligned}$$

If $v_\phi \in [\underline{v}, \bar{v}]$ then each term in the sum is non-negative by the local sector constraints and $\lambda \geq 0$. \square

In order to apply the local sector bounds in the stability analysis, we must first compute the bounds $\underline{v}, \bar{v} \in \mathbb{R}^{n_\phi}$ on the activation input v_ϕ . The process to compute the bounds is briefly discussed here with more details provided in [9], [19]. The first step is to find the smallest hypercube that bounds the state constraint set: $X \subset \{x : \underline{x} \leq x \leq \bar{x}\}$. Therefore, w^0 (defined in (4a)) is bounded by $\underline{w}^0 = \underline{x}$ and $\bar{w}^0 = \bar{x}$. Define $c = \frac{1}{2}(\bar{w}^0 + \underline{w}^0)$, $r = \frac{1}{2}(\bar{w}^0 - \underline{w}^0)$, and denote y^\top as the i^{th} row of W^1 . Then the first activation input $v^1 = W^1 w^0$ is bounded by $[\underline{v}^1, \bar{v}^1]$, where

$$\bar{v}_i^1 = y^\top c + \sum_{j=1}^{n_0} |y_j r_j|, \quad (15a)$$

$$\underline{v}_i^1 = y^\top c - \sum_{j=1}^{n_0} |y_j r_j|. \quad (15b)$$

If the activation functions ϕ^1 are monotonically non-decreasing, then the first activation output w^1 is bounded by $\underline{w}^1 = \phi^1(\underline{v})$ and $\bar{w}^1 = \phi^1(\bar{v})$. This process can be propagated through all layers of the NN to obtain the bounds $\underline{v}, \bar{v} \in \mathbb{R}^{n_\phi}$ for the activation input v_ϕ . The remainder of the paper will assume these local sector bounds have been computed as briefly summarized in the following property.

Property 1. Let the state constraint set X and its corresponding activation input bounds \underline{v}, \bar{v} be given. There exist

$\alpha_\phi, \beta_\phi \in \mathbb{R}^{n_\phi}$ such that ϕ satisfies the local sector for all $v_\phi \in [\underline{v}, \bar{v}]$.

D. Lyapunov Condition

This section uses a Lyapunov function and the local sector to compute an inner approximation for the ROA of the feedback system of G and π .

Theorem 1. Consider the feedback system of plant G in (2) and NN π in (4) with equilibrium point $x_* = 0_{n_G \times 1}$, and the state constraint set X . Let $\bar{v}, \underline{v}, \alpha_\phi, \beta_\phi \in \mathbb{R}^{n_\phi}$ be given vectors satisfying Property 1 for the NN and the set X . Denote the i^{th} row of the matrix H by H_i^\top and define matrices

$$R_V := \begin{bmatrix} I_{n_G} & 0_{n_G \times n_\phi} \\ N_{ux} & N_{uw} \end{bmatrix}, \text{ and } R_\phi := \begin{bmatrix} N_{vx} & N_{vw} \\ 0_{n_\phi \times n_G} & I_{n_\phi} \end{bmatrix}.$$

If there exists a matrix $P \in \mathbb{S}_{++}^{n_G}$, and vector $\lambda \in \mathbb{R}^{n_\phi}$ with $\lambda \geq 0$ such that

$$R_V^\top \begin{bmatrix} A_G^\top P A_G - P & A_G^\top P B_G \\ B_G^\top P A_G & B_G^\top P B_G \end{bmatrix} R_V + R_\phi^\top \begin{bmatrix} -2A_\phi B_\phi \Lambda & (A_\phi + B_\phi) \Lambda \\ (A_\phi + B_\phi) \Lambda & -2\Lambda \end{bmatrix} R_\phi < 0, \quad (16a)$$

$$\begin{bmatrix} h_i^2 & H_i^\top \\ H_i & P \end{bmatrix} \geq 0, \quad i = 1, \dots, n_X, \quad (16b)$$

where $\Lambda = \text{diag}(\lambda)$, then: (i) the feedback system consisting of G and π is locally asymptotically stable around x_* , and (ii) the set $\mathcal{E}(P)$, defined by Equation 1, is an inner-approximation to the ROA.

Proof. By Schur complements, (16b) is equivalent to:

$$H_i^\top P^{-1} H_i \leq h_i^2, \quad i = 1, \dots, n_X. \quad (17)$$

It follows from Lemma 1 in [20] that:

$$\mathcal{E}(P) \subseteq \{x \in \mathbb{R}^{n_G} : -h_i \leq H_i^\top x \leq h_i, \quad i = 1, \dots, n_X\} = X.$$

To summarize, feasibility of (16b) verifies that if $x(k) \in \mathcal{E}(P) \subseteq X$ then $v_\phi(k) \in [\underline{v}, \bar{v}]$ and hence the local sector conditions are valid.

Next, left / right multiply the LMI (16a) by any (non-zero) $[x(k)^\top \quad w_\phi(k)^\top]$ and its transpose to obtain

$$\begin{bmatrix} x(k) \\ u(k) \end{bmatrix}^\top \begin{bmatrix} A_G^\top P A_G - P & A_G^\top P B_G \\ B_G^\top P A_G & B_G^\top P B_G \end{bmatrix} \begin{bmatrix} x(k) \\ u(k) \end{bmatrix} + \begin{bmatrix} v_\phi(k) \\ w_\phi(k) \end{bmatrix}^\top \begin{bmatrix} -2A_\phi B_\phi \Lambda & (A_\phi + B_\phi) \Lambda \\ (A_\phi + B_\phi) \Lambda & -2\Lambda \end{bmatrix} \begin{bmatrix} v_\phi(k) \\ w_\phi(k) \end{bmatrix} < 0.$$

Define $V(x) := x^\top P x$ and use (2) to show:

$$V(x(k+1)) - V(x(k)) + \begin{bmatrix} v_\phi(k) \\ w_\phi(k) \end{bmatrix}^\top \begin{bmatrix} -2A_\phi B_\phi \Lambda & (A_\phi + B_\phi) \Lambda \\ (A_\phi + B_\phi) \Lambda & -2\Lambda \end{bmatrix} \begin{bmatrix} v_\phi(k) \\ w_\phi(k) \end{bmatrix} < 0. \quad (18)$$

As noted above, $x(k) \in \mathcal{E}(P)$ implies the local sector $[\alpha_\phi, \beta_\phi]$. Then, by Lemma 1, the final term in (18) is ≥ 0 and it follows from a Lyapunov argument, e.g. Theorem 4.1 in [21], that x_* is an asymptotically stable equilibrium point. Using the proof by contradiction argument from [22, Theorem 1], we

can show that $\mathcal{E}(P)$ is an invariant set: if $x(0) \in \mathcal{E}(P)$, then $x(k) \in \mathcal{E}(P) \forall k \geq 0$, and thus $\mathcal{E}(P)$ is an inner approximation of the ROA. \square

The Lyapunov condition (16a) is convex in P and λ if the weight matrix N is given, and thus we can efficiently compute the ROA inner-estimates. However, this condition is computationally intractable for NN controller synthesis, as it is nonconvex if we search over N , P , and λ simultaneously.

IV. CONVEX STABILITY AND SAFETY CONDITIONS

In [11], [12], α_ϕ is set to zero in order to formulate convex constraints. This restriction is too coarse for stability analysis of NN controlled systems. Instead, we perform a loop transformation as shown in Figure 4 to convexify the stability condition without having restrictions on α_ϕ and β_ϕ .

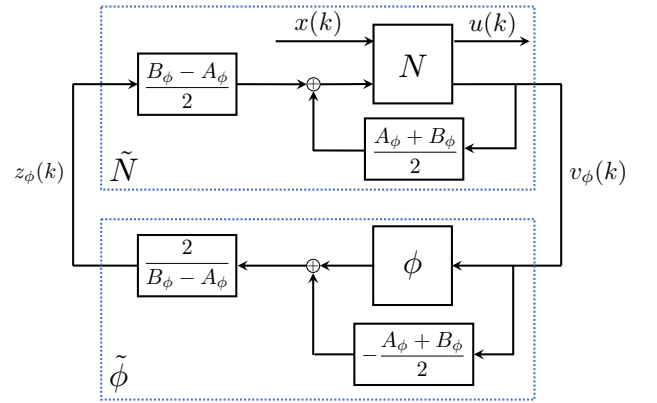


Fig. 4: Loop transformation. If ϕ is in the sector $[\alpha_\phi, \beta_\phi]$, then $\tilde{\phi}$ is in the sector $[-1_{n_\phi \times 1}, 1_{n_\phi \times 1}]$.

A. Loop transformation

Through loop transformation [5], we obtain a new representation of the NN controller, which is equivalent to (10),

$$\begin{bmatrix} u(k) \\ v_\phi(k) \end{bmatrix} = \tilde{N} \begin{bmatrix} x(k) \\ z_\phi(k) \end{bmatrix}, \quad (19a)$$

$$z_\phi(k) = \tilde{\phi}(v_\phi(k)), \quad (19b)$$

where \tilde{N} and $\tilde{\phi}$ are defined in Figure 4. Here, we also partition \tilde{N} compatibly with the inputs (x, z_ϕ) and outputs (u, v_ϕ)

$$\tilde{N} = \begin{bmatrix} \tilde{N}_{ux} & \tilde{N}_{uz} \\ \tilde{N}_{vx} & \tilde{N}_{vz} \end{bmatrix}. \quad (20)$$

The loop transformation normalizes the nonlinearity $\tilde{\phi}$ to lie in the sector $[-1_{n_\phi \times 1}, 1_{n_\phi \times 1}]$. As a result, $\tilde{\phi}$ satisfies the sector quadratic constraint[†] for any $\Lambda = \text{diag}(\lambda)$ with $\lambda \geq 0$:

$$\begin{bmatrix} v_\phi(k) \\ z_\phi(k) \end{bmatrix}^\top \begin{bmatrix} \Lambda & 0 \\ 0 & -\Lambda \end{bmatrix} \begin{bmatrix} v_\phi(k) \\ z_\phi(k) \end{bmatrix} \geq 0, \quad \forall v_\phi \in [\underline{v}, \bar{v}]. \quad (21)$$

[†]Note that when substituting $A_\phi = I_{n_\phi}$ and $B_\phi = I_{n_\phi}$ in (14), we get

$$\begin{bmatrix} v_\phi(k) \\ z_\phi(k) \end{bmatrix}^\top \begin{bmatrix} 2\Lambda & 0 \\ 0 & -2\Lambda \end{bmatrix} \begin{bmatrix} v_\phi(k) \\ z_\phi(k) \end{bmatrix} \geq 0, \quad \forall v_\phi \in [\underline{v}, \bar{v}].$$

The factor of 2 is absorbed into Λ in (21).

The input to N is transformed by the following algebraic equation:

$$w_\phi(k) = \frac{B_\phi - A_\phi}{2} z_\phi(k) + \frac{A_\phi + B_\phi}{2} v_\phi(k). \quad (22)$$

The transformed matrix \tilde{N} can be computed by combining this relation with (10a). Substituting (22) into (10a) we obtain

$$u(k) = N_{ux}x(k) + C_1 z_\phi(k) + C_2 v_\phi(k), \quad (23)$$

$$v_\phi(k) = N_{vx}x(k) + C_3 z_\phi(k) + C_4 v_\phi(k), \quad (24)$$

$$\text{where } C_1 = N_{uw} \frac{B_\phi - A_\phi}{2}, \quad C_2 = N_{uw} \frac{A_\phi + B_\phi}{2}, \\ C_3 = N_{vw} \frac{B_\phi - A_\phi}{2}, \quad C_4 = N_{vw} \frac{A_\phi + B_\phi}{2}.$$

The expression for $v_\phi(k)$ can be solved from (24):

$$v_\phi(k) = (I - C_4)^{-1} N_{vx}x(k) + (I - C_4)^{-1} C_3 z_\phi(k). \quad (25)$$

Substituting (25) into (23) yields

$$u(k) = (N_{ux} + C_2(I - C_4)^{-1} N_{vx})x(k) \\ + (C_1 + C_2(I - C_4)^{-1} C_3)z_\phi(k). \quad (26)$$

Matching (25) and (26) with (19a) we get

$$\tilde{N} = \begin{bmatrix} N_{ux} + C_2(I - C_4)^{-1} N_{vx} & C_1 + C_2(I - C_4)^{-1} C_3 \\ (I - C_4)^{-1} N_{vx} & (I - C_4)^{-1} C_3 \end{bmatrix}.$$

Thus, \tilde{N} is a function of N denoted concisely as $\tilde{N} = f(N)$. It is important to note that \tilde{N} depends on N both directly, and also indirectly through its dependence on (A_ϕ, B_ϕ) . Specifically, suppose both N and a hypercube state bound (\underline{x}, \bar{x}) are given. Then \tilde{N} is constructed by: (i) propagating (\underline{x}, \bar{x}) through the NN to compute bounds \bar{v}, \underline{v} on the activation inputs, (ii) computing local sector bounds (A_ϕ, B_ϕ) consistent with the activation bounds, and (iii) performing the steps in this section to compute \tilde{N} from (N, A_ϕ, B_ϕ) .

B. Stability condition after loop transformation

Similar to the original Lyapunov condition (16a), the new Lyapunov condition for the feedback system of G in (2) and NN in (19) can be written as

$$\tilde{R}_V^\top \begin{bmatrix} A_G^\top P A_G - P & A_G^\top P B_G \\ B_G^\top P A_G & B_G^\top P B_G \end{bmatrix} \tilde{R}_V + \tilde{R}_\phi^\top \begin{bmatrix} \Lambda & 0 \\ 0 & -\Lambda \end{bmatrix} \tilde{R}_\phi < 0, \quad (27)$$

where $\Lambda := \text{diag}(\lambda)$, and

$$\tilde{R}_V = \begin{bmatrix} I_{n_G} & 0_{n_G \times n_\phi} \\ \tilde{N}_{ux} & \tilde{N}_{uz} \end{bmatrix}, \quad \tilde{R}_\phi = \begin{bmatrix} \tilde{N}_{vx} & \tilde{N}_{vz} \\ 0_{n_\phi \times n_G} & I_{n_\phi} \end{bmatrix}. \quad (28)$$

Lemma 2. Consider the feedback system of G in (2) and NN in (10) with the state constraint set X . If there exist a matrix $P \in \mathbb{S}_{++}^{n_G}$, and vector $\lambda \in \mathbb{R}^{n_\phi}$ such that (27) (where $\tilde{N} = f(N)$) and (16b) hold, then: (i) the feedback system consisting of G in (2) and NN in (10) is locally asymptotically stable around x_* , and (ii) the set $\mathcal{E}(P)$ is a ROA inner-approximation for it.

Proof. It follows from the assumption that (27) and (16b) hold that the feedback system of G in (2) and NN in (19) is locally

asymptotically stable around x_* , and $\mathcal{E}(P)$ is its ROA inner-approximation. Since the representations (10) and (19) of NN are equivalent, the feedback system of G in (2) and NN in (10) is identical to the feedback system of G in (2) and NN in (19). As a result, the feedback system consisting of G in (2) and NN in (10) is locally asymptotically stable around x_* , and the set $\mathcal{E}(P)$ is a ROA inner-approximation for it. \square

The new Lyapunov condition (27) is convex in P and Λ using $\tilde{N} = f(N)$, where N is given. To incorporate the stability condition in the IL process, we will proceed by treating $\tilde{N} \in \mathbb{R}^{(n_u+n_\phi) \times (n_G+n_\phi)}$ as a decision variable along with P and Λ , and try to derive a stability condition that is convex in (P, Λ, \tilde{N}) . Substitute (28) into (27) to obtain

$$[\star]^\top \begin{bmatrix} P & 0 \\ 0 & \Lambda \end{bmatrix} \begin{bmatrix} A_G + B_G \tilde{N}_{ux} & B_G \tilde{N}_{uz} \\ \tilde{N}_{vx} & \tilde{N}_{vz} \end{bmatrix} - \begin{bmatrix} P & 0 \\ 0 & \Lambda \end{bmatrix} < 0. \quad (29)$$

Applying Schur complements, we can write the equivalent condition

$$\begin{bmatrix} P & 0 & A_G^\top + \tilde{N}_{ux}^\top B_G^\top & \tilde{N}_{vx}^\top \\ 0 & \Lambda & \tilde{N}_{uz}^\top B_G^\top & \tilde{N}_{vz}^\top \\ A_G + B_G \tilde{N}_{ux} & B_G \tilde{N}_{uz} & P^{-1} & 0 \\ \tilde{N}_{vx} & \tilde{N}_{vz} & 0 & \Lambda^{-1} \end{bmatrix} > 0, \quad (30)$$

and $P > 0$, $\Lambda > 0$. Now (30) is linear in \tilde{N} , but still nonconvex in P and Λ . Multiplying (30) on the left and right by $\text{diag} \left(\begin{bmatrix} P^{-1} & 0 \\ 0 & \Lambda^{-1} \end{bmatrix}, I_{n_G+n_\phi} \right)$ we obtain

$$\begin{bmatrix} Q_1 & 0 & Q_1 A_G^\top + L_1^\top B_G^\top & L_3^\top \\ 0 & Q_2 & L_2^\top B_G^\top & L_4^\top \\ A_G Q_1 + B_G L_1 & B_G L_2 & Q_1 & 0 \\ L_3 & L_4 & 0 & Q_2 \end{bmatrix} > 0, \quad (31)$$

where

$$Q_1 = P^{-1} > 0, \quad Q_2 = \Lambda^{-1} > 0,$$

$$L_1 = \tilde{N}_{ux} Q_1, \quad L_2 = \tilde{N}_{uz} Q_2, \quad L_3 = \tilde{N}_{vx} Q_1, \quad L_4 = \tilde{N}_{vz} Q_2.$$

The stability condition (31) is convex in the decision variables $(Q_1, Q_2, L_1, \dots, L_4)$, where $Q_1 \in \mathbb{S}_{++}^{n_G}$, $Q_2 \in \mathbb{S}_{++}^{n_\phi}$ and Q_2 is a diagonal matrix, $L_1 \in \mathbb{R}^{n_u \times n_G}$, $L_2 \in \mathbb{R}^{n_u \times n_\phi}$, $L_3 \in \mathbb{R}^{n_\phi \times n_G}$, and $L_4 \in \mathbb{R}^{n_\phi \times n_\phi}$. Variables (P, Λ, \tilde{N}) that satisfy the Lyapunov condition (27) can be recovered using the computed $(Q_1, Q_2, L_1, \dots, L_4)$ through the following equations: $P = Q_1^{-1}$, $\Lambda = Q_2^{-1}$, and

$$\tilde{N} = \begin{bmatrix} \tilde{N}_{ux} & \tilde{N}_{uz} \\ \tilde{N}_{vx} & \tilde{N}_{vz} \end{bmatrix} = L Q^{-1}, \quad (32)$$

$$\text{where } Q := \begin{bmatrix} Q_1 & 0 \\ 0 & Q_2 \end{bmatrix}, \quad \text{and } L := \begin{bmatrix} L_1 & L_2 \\ L_3 & L_4 \end{bmatrix}. \quad (33)$$

Thus, the convex stability condition (31) allows us to search over P , Λ , and \tilde{N} simultaneously.

To enforce the safety condition $(x(k) \in X \forall k \geq 0)$ of the system, convex constraints on Q_1 are imposed:

$$H_i^\top Q_1 H_i \leq h_i^2, \quad i = 1, \dots, n_X, \quad (34)$$

which is derived directly from (16b) by Schur complements, and using $Q_1 = P^{-1}$. Again, this ensures $\mathcal{E}(Q_1^{-1}) \subseteq X$.

We denote the LMIs (31), (34) with $Q_1 > 0$ and $Q_2 > 0$ altogether as $\text{LMI}(Q, L) > 0$. Later in this paper, these LMIs will be incorporated in the IL process to robustify the NN controller.

Remark 2. *Note that model uncertainties and nonlinearities are not considered in the paper. They can be incorporated in the Lyapunov condition (16a) or (27) using integral quadratic constraints as in [9]. However, to derive convex stability conditions, only limited types of uncertainties may be incorporated.*

V. SAFE IMITATION LEARNING ALGORITHM

Given state and control data pairs from the expert demonstrations, we use a loss function $\mathcal{L}(N)$ to train NN controllers with weights N to match the data. Common choices of the loss function includes mean squared error, absolute error, and cross-entropy. In general, $\mathcal{L}(N)$ is non-convex in N . We propose the next optimization to solve the safe IL problem,

$$\min_{N, Q, L} \eta_1 \mathcal{L}(N) - \eta_2 \log \det(Q_1) \quad (35a)$$

$$\text{s.t. } \text{LMI}(Q, L) > 0 \quad (35b)$$

$$f(N)Q = L \quad (35c)$$

where Q and L are defined in (33). The optimization has separate objectives. The cost function (35a) combines the IL loss function with a term that attempts to increase the volume of $\mathcal{E}(Q_1^{-1})$ (which is proportional to $\det(Q_1)$). The parameters $\eta_1, \eta_2 > 0$ reflect the relative importance between imitation learning accuracy and size of the robustness margin. The optimization has two sets of decision variables: N and (Q, L) . The former is involved in mimicking the expert behaviour, and the latter are involved in the stability and safety constraints (35b). The two sets of variables are connected through the equality constraint (35c). Note that (35c) is equivalent to $f(N) = LQ^{-1}$, and the term on the right-hand side equals to \tilde{N} from (32). Therefore, (35c) essentially means that the first set of decision variable N , after being transformed by the nonlinear function f , satisfies the stability and safety constraints.

Similar to [11], we use the alternating direction method of multipliers (ADMM) algorithm to solve this type of constrained learning problem. We first define an augmented loss function

$$\mathcal{L}_a(N, Q, L, Y) = \eta_1 \mathcal{L}(N) - \eta_2 \log \det(Q_1) + \text{tr}(Y^\top \cdot (f(N)Q - L)) + \frac{\rho}{2} \|f(N)Q - L\|_F^2, \quad (36)$$

where $\|\cdot\|_F$ is the Frobenius norm, $Y \in \mathbb{R}^{(n_u+n_\phi) \times (n_G+n_\phi)}$ is the Lagrange multiplier, and $\rho > 0$ is the regularization parameter typically affecting the convergence rate of ADMM. The ADMM algorithm takes the following form

1. N -update

$$N^{k+1} = \arg \min_N \mathcal{L}_a(N, Q^k, L^k, Y^k) \quad (37a)$$

2. (Q, L) -update

$$(Q^{k+1}, L^{k+1}) = \arg \min_{Q, L} \mathcal{L}_a(N^{k+1}, Q, L, Y^k) \quad \text{s.t. } \text{LMI}(Q, L) > 0 \quad (37b)$$

3. Y -update: If $\|f(N^{k+1})Q^{k+1} - L^{k+1}\|_F \leq \sigma$, where $\sigma > 0$ is the stopping tolerance, then the algorithm has converged, and we have found a solution to (35), so terminate the algorithm. Otherwise, update Y and return to step 1.

$$Y^{k+1} = Y^k + \rho (f(N^{k+1})Q^{k+1} - L^{k+1}) \quad (37c)$$

Step 1 can be solved using gradient based algorithm, e.g., stochastic gradient descent, or ADAM [23]. The optimization in Step 2 is convex, and can be solved effectively using SDP solvers, e.g., MOSEK. The variable Y in Step 3 accumulates the deviation from the constraint (35c) as in integral control.

Since the loss function $\mathcal{L}(N)$ is generally nonconvex, and the constraint (35c) is also nonconvex, the proposed ADMM does not have the guarantee to converge to the global optima. However, any converged solution (even local optima) provides a safe NN controller with stability and safety guarantees.

VI. EXAMPLES

In the following examples, Step 1 of ADMM algorithm is implemented on Tensorflow, and solved by ADAM [23]. Step 2 is formulated using CVX, and is solved by MOSEK. The mean squared error is chosen as the loss function $\mathcal{L}(N)$.

A. Inverted pendulum

Consider an inverted pendulum with mass $m = 0.15$ kg, length $l = 0.5$ m, and friction coefficient $\mu = 0.5$ Nms/rad. The discretized and linearized dynamics are:

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \end{bmatrix} = \begin{bmatrix} 1 & \delta \\ \frac{g\delta}{l} & 1 - \frac{\mu\delta}{ml^2} \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{\delta}{ml^2} \end{bmatrix} u(k),$$

where the states x_1, x_2 represent the angular position (rad) and velocity (rad/s), u is the control input (Nm), and $\delta = 0.02$ s is the sampling time. The state constraint set is $X = [-2.5, 2.5] \times [-6, 6]$. To generate state and control data pairs for IL, we design an explicit model predictive controller (MPC) to serve as the expert. By fitting a NN controller to the explicit MPC controller, we can expedite the evaluation of controllers during run-time [24]–[26]. In this example, besides a NN controller, we will also provide its associated ROA inner-approximation that guarantees stability and safety. The NN controller is parameterized by a 2-layer, feedforward NN with $n_1 = n_2 = 16$ and tanh as the activation function for both layers. Take $\rho = 1$, $\eta_1 = 200$, and $\eta_2 = 10$. The ADMM algorithm is terminated after 16 iterations, and $\|f(N) - LQ^{-1}\|_F = 0.094$.

In Figure 5, the ROA inner-approximation (shown with a blue ellipsoid) is tightly contained by the state constraint set X (shown with an orange rectangle), which guarantees the safety of the system. Figure 6 shows the learned NN controller with a blue surface, and state and control data pairs from expert demonstrations with orange stars. Figure 7 shows the mesh plots of $f(N)$ and LQ^{-1} after 16 iterations of ADMM.

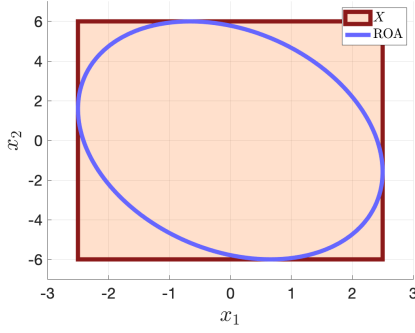


Fig. 5: A ROA inner-approximation and state constraint set X of the inverted pendulum

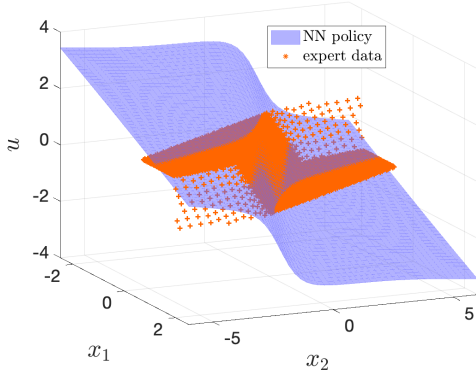


Fig. 6: NN controller vs. expert data of the inverted pendulum

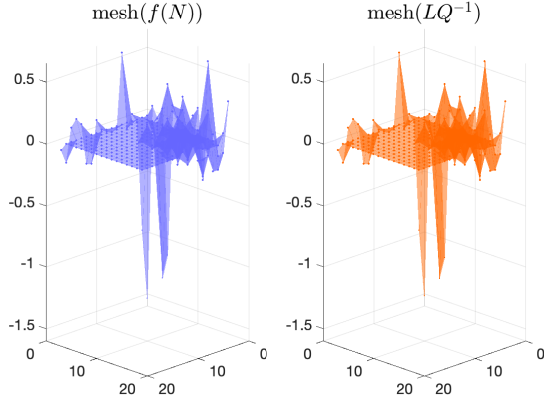


Fig. 7: $f(N)$ vs. LQ^{-1} after running 16 iterations of the ADMM algorithm of the inverted pendulum

B. Generic Transport Model

The Generic Transport Model (GTM) is a 5.5% scale commercial aircraft. Linearizing and discretizing the longitudinal dynamics given in [27] with sampling time $\delta = 0.02$ s yields:

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \end{bmatrix} = \begin{bmatrix} 0.935 & 0.019 \\ -0.907 & 0.913 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} + \begin{bmatrix} -0.006 \\ -1.120 \end{bmatrix} u(k),$$

where the states x_1, x_2 represent angle of attack (rad), and pitch rate (rad/s), and the control u represents the elevator deflection (rad). Take the state constraint set as $X = [-2, 2] \times [-3, 3]$. In this example, we design an LQR controller to produce expert data. The NN controller is again parameterized

by a 2-layer, feedforward NN with $n_1 = n_2 = 16$ and tanh as the activation function for both layers. In this example, we will show how the parameter η_2 affects the result. To do so, two experiments are carried out using two sets of parameters ($\rho = 1, \eta_1 = 100, \eta_2 = 5$) and ($\rho = 1, \eta_1 = 100, \eta_2 = 20$), meaning that we care more about the size of the ROA inner-approximation, and less about the IL accuracy in the second experiment than we do in the first experiment. In both experiments, the ADMM algorithm is terminated in 20 iterations.

The ROA inner-approximations of the NN controllers from the two experiments are shown in Figure 8. The one computed with $\eta_2 = 5$ is shown with a magenta ellipsoid, and the one computed with $\eta_2 = 20$ is shown with a blue ellipsoid. First, it is important to notice that both NN controllers' ROA inner-approximations are larger than that of the expert's LQR controller (shown with a dashed gray ellipsoid), thanks to the second term in the cost function (35a), which enhances the robustness of IL. Also, as expected, the ROA inner-approximation of the NN controller with $\eta_2 = 20$ is larger than that with $\eta_2 = 5$, since a larger η_2 leads to a larger ROA inner-approximation. However, the larger ROA inner-approximation comes at the cost of less accurate regression to the expert data. As shown in Figure 9, the mesh plot of the NN controller with $\eta_2 = 20$ (shown with a blue surface) is more off from the expert data (shown with orange stars) than that with $\eta_2 = 5$ (shown with a magenta surface).

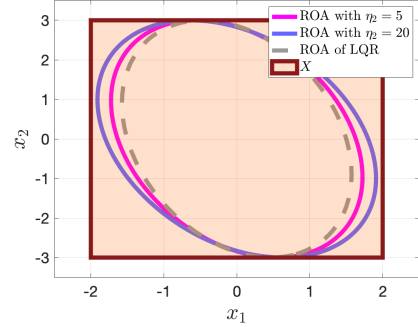


Fig. 8: ROA inner-approximations and state constraint set X of GTM

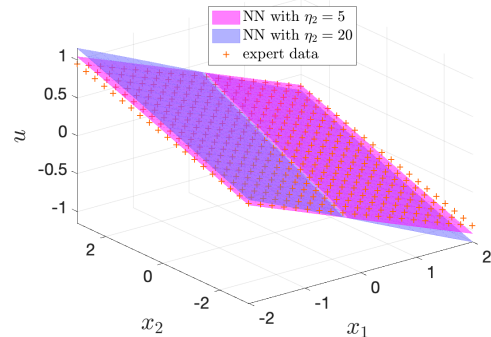


Fig. 9: NN controllers vs. expert data of GTM

VII. CONCLUSIONS

In this paper, we present an imitation learning algorithm with local stability and safety guarantees for LTI systems. First, we convexify the stability and safety conditions for NN controlled systems from [9] using loop transformation. Second, we formulate the safe imitation learning problem as an optimization problem that trades off between the imitation learning accuracy, and size of the stability margin, while enforcing the stability and safety constraints. We propose an ADMM based algorithm to solve the constrained optimization. Finally, we illustrate the method on an inverted pendulum example and an aircraft example. Future work will address incorporation of the stability and safety conditions in the reinforcement learning process.

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